# Distributed Control Policies for Localization of Large Disturbances in Urban Traffic Networks

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*Abstract*— **We present a distributed control strategy to localize and attenuate traffic jams caused by large disturbances in urban transportation networks. The control policy is distributed in the sense that it uses those traffic lights in the vicinity of the jammed area. We model urban traffic using a discrete fluid-like model which is then reduced to a hybrid dynamical system with binary control inputs. For this system, we define notions of local control and localizability of disturbances. We then present a control strategy that uses only local controllers to attenuate traffic jams. This control design is formulated as a mixed-integer linear program (MILP) whose solution provides traffic light schedules. We show that the feasibility of this MILP is both necessary and sufficient for localizability. Finally, we illustrate this design on a test urban transportation network.**

## I. INTRODUCTION

In the United States alone, propagating traffic jams lead to billions of wasted man-hours and several million gallons of wasted fuel every year [1]. In urban traffic networks, fixed-time or partially real-time traffic light scheduling policies such as SCOOT [2] and SCATS [3] have been proposed to reduce congestion and enhance throughput, and are widely used in cities around the world [4]. With the development of more detailed traffic models, Model Predictive Control (MPC)-based strategies have been developed for coordinated signal control [4]- [8] in urban networks. In [9], model-based verification and synthesis tools have been used to control urban traffic networks from temporal logic [10] specifications that typically pertain to objectives such as avoidance of congestion and temporal constraints on traffic signals.

While the controllers designed by these approaches are robust to small uncertainties and disturbances, large disturbances corresponding to temporary demand surges in peak hours or unmodeled driver behavior may cause traffic jams that quickly propagate through the entire network. In this scenario, it is desirable to obtain control policies that bring the system back from this congested state to a desired operating regime, ideally within a specified deadline and using only control signals that are local to the vicinity of the largely disturbed area. In our earlier work [11], we considered this problem of localized control [12] for freeway networks; however, the results were restricted to simple network topologies without merge and diverge intersections, and assumed full state feedback with continuous actuation. In this paper, we solve this problem for urban networks, with no restrictions on the allowed network topology and more realistic actuation via discrete traffic light schedules.

We consider a discrete-time fluid-like model similar to the widely-used CTM [13] to describe urban transportation networks, with traffic signal schedules at intersections as actuators. We reformulate this model as a hybrid system and assume that a scheduling policy of traffic lights, such as that proposed in [14], is in place to ensure that the system trajectories remain in a desired set (which may, for example, correspond to free flow) in the presence of small disturbances. When large disturbances like peak-hour demand surges occur, a fixed-time schedule such as that proposed in [14] is no longer sufficient to maintain the system in this set, leading to a traffic jam. We then formulate a mixed-integer linear program (MILP), whose solution provides a control policy that returns the system to the desired operating set within a specified time period using controllers in the local neighborhood of the jammed area. This control strategy further ensures that the jam does not propagate beyond this local area. We show that the feasibility of the proposed MILP is necessary and sufficient to ensure localization.

The contributions of this paper are as follows. First, we present a control policy that localizes jams to small sections of the transportation network and prevents their propagation using only local controllers in the immediate vicinity of the jammed area. In contrast to centralized MPC-based schemes such as [15], our design explicitly imposes the condition that jams do not spread beyond a specified radius. Second, our control synthesis takes into account the hybrid nature of the dynamics and discrete actuation using an MILP formulation. This was not accounted for in earlier disturbance localization schemes like [11].

**Notation:** Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ denote the set of nonnegative real numbers. Let $\mathbb{R}_+^N$ denote the set of all nonnegative $N$-vectors. For two vectors $x, y \in \mathbb{R}^N$, we have $x \prec y \iff x_i < y_i$ and $x \preceq y \iff x_i \leq y_i$, $i = 1, \ldots, N$. Let $\mathbb{N}$ denote the set of all natural numbers including zero and let $1_N$ denote the $N$-vector of all ones.

## II. MODEL

We use a discrete-time fluid-like model, similar to that proposed in [16] to describe the dynamics of urban transportation networks. However, our model differs from [16] in that we assume a non-first-in-first-out (non-FIFO) model to describe driver behavior at intersections, where drivers

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Vijay Gupta and Sivaranjani S were partially funded by the AFOSR grant FA 9550-15-1-0186 and NSF grants ECCS-1550016 and CNS 1239224. Calin Belta and Sadra Sadraddini were partially funded by NSF grants CPS-1446151 and CMMI-1400167. Sivaranjani S was partially supported by the Schlumberger Foundation Faculty for the Future Fellowship.

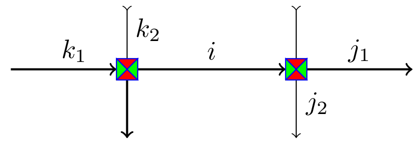


Fig. 1. A simple traffic network with 7 cells and 2 intersections controlled by traffic lights.

choosing a particular route that is congested do not block drivers choosing other routes at intersections.

*Network Topology*

The transportation network is defined as a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, whose edges $\mathcal{E}$ represent cells and nodes $\mathcal{V}$ represent intersections. In this model, each cell $i \in \mathcal{E}$ corresponds to a section of a one-way road from tail node $\tau_i$ toward head node $h_i$, where $\tau_i, h_i \in \mathcal{V} \cup \emptyset$ (and $\emptyset$ denotes the world outside the network, typically another network). The total number of cells in the network is denoted by $N = |\mathcal{E}|$. For each cell $i \in \mathcal{E}$, we define the following sets:

$$\begin{aligned} &\text{Outgoing Cells: } \mathcal{E}_i^{out} := \{j \in \mathcal{E} : \tau_j = h_i\}, \\ &\text{Incoming Cells: } \mathcal{E}_i^{in} := \{k \in \mathcal{E} : h_k = \tau_i\}. \end{aligned} \tag{1}$$

As an example, a section of a transportation network is illustrated in Fig. 1, where we have $\mathcal{E}_i^{in} = \{k_1, k_2\}$ and $\mathcal{E}_i^{out} = \{j_1, j_2\}$.

*Definition* 1 (Adjacency Matrix). The network adjacency matrix of a transportation network $(\mathcal{V}, \mathcal{E})$ is defined as $A \in \mathbb{R}^{N \times N}$ where $A = [A_{ij}], i, j \in \{1, \ldots, N\}$ with

$$A_{ij} = \begin{cases} 1 & j = i, \\ 1 & j \neq i,\ j \in \mathcal{E}_i^{in} \cup \mathcal{E}_i^{out}, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

*System Dynamics*

We denote the number of vehicles in cell $i$ at time $t \in \mathbb{Z}_{\geq 0}$ by $\rho_i(t)$. The system state is defined as the $N$-vector $\rho(t)$, obtained by stacking the density in each cell, that is, $\rho(t) \triangleq [\rho_1(t) \ldots \rho_N(t)]'$. To each cell $i$, we assign piecewise affine sending and receiving flow functions, denoted by $S_i(t)$ and $R_i(t)$ respectively, defined as

$$S_i(t) = u_i(t) \min\{M_i, v_{f,i} \rho_i(t)\}, \tag{3}$$
$$R_i(t) = \min\{M_i, \hat{v}_{c,i}(C_i - \rho_i(t))\}, \tag{4}$$

where $v_{f,i} \in \mathbb{R}_+$, $\hat{v}_{c,i} \in \mathbb{R}_+$, $C_i$, and $M_i$ are the free speed, congested speed, jamming density and maximum outflow (in one time step) of cell $i$ respectively, and $u_i(t) \in \{0, 1\}$ is the control input at time $t$, where 0 stands for a red light and 1 denotes a green light. We then have $\rho(t) \in \mathcal{P}, \forall t \in \mathbb{Z}_{\geq 0}$, where $\mathcal{P} = \prod_{i=1}^{N} [0, C_i]$. We denote the vector representation of the control inputs by $u(t) \in \mathcal{U}, \mathcal{U} \subset \{0, 1\}^N$, where $\mathcal{U}$ is the set of all admissible combinations of traffic lights. Typically, in accordance with traffic conventions, we disallow simultaneous green lights for two cells that merge into a common intersection in different directions, $i_1$ and $i_2$, by imposing an integer constraint $u_{i_1}(t) + u_{i_2}(t) \leq 1$. Let $r_{i,j} \in \mathbb{R}_+$, referred to as the *turning ratio*, be the fraction of vehicles in cell $i$ choosing to turn into cell $j \in \mathcal{E}_i^{out}$, with $\sum_{j \in \mathcal{E}_i^{out}} r_{i,j} = 1$. For simplicity, we assume that the turning ratios are constant for the remainder of this paper. However, they can be assumed to lie in any known interval for our design. For cell $i \in \mathcal{E}$, the inflow is:

$$Q_{i,in}(\rho(t)) = \sum_{k \in \mathcal{E}_i^{in}} \alpha_{k,i}^u \min\{\lambda_i + S_k(t) + w_i(t), R_i(t)\}, \tag{5}$$

where $\lambda_i$ is the constant baseline exogenous flow into cell $i$ and $w_i(t)$ denotes an additive disturbance acting on cell $i$ at time $t$. The *capacity ratio* $\alpha_{k,i}^u$ represents the fraction of the vacant space in cell $k$ available to cell $i$ when the control signal is $u$. Note that we have, $\forall i \in \mathcal{E}$, $\sum_{k \in \mathcal{E}_i^{in}} \alpha_{k,i}^u = 1, \forall u \in \mathcal{U}$. For simple gridded networks of one-way roads, we often assume that $\alpha_{k,i}^u = 1, \forall i, k, u$.

Let $w(t) = [w_1(t) \ldots w_N(t)]'$ be the $N$-vector of the disturbances in all cells. These disturbances are added to account for unmodeled changes in the density profile due to surges on the baseline demand during peak hours. We assume that these disturbances are in a bounded set, that is, $w(t) \in \mathcal{W} \subset \mathbb{R}^N$ for all time $t$, where $\mathcal{W} = \{w(t) : 0 \preceq w(t) \preceq w^*, w^* \in \mathbb{R}^N\}$. The outflow of cell $i$ is given by

$$Q_{i,out}(\rho(t)) = \sum_{j \in \mathcal{E}_i^{out}} r_{i,j} \min\{S_i(t), R_j(t)\}. \tag{6}$$

The flow model above corresponds to a simple non-first-in-first-out (non-FIFO) policy at diverge intersections, where congestion in an outgoing cell at the intersection does not cause congestion in the incoming cells. This assumption is valid for large roads with multiple turning lanes. We note that more sophisticated non-FIFO models have also been proposed to accurately capture driver choices [17] [18].

Finally, the evolution of the density in cell $i$ is described by the following difference equation

$$\rho_i(t+1) = \rho_i(t) + Q_{i,in}(\rho(t)) - Q_{i,out}(\rho(t)),\ i \in \mathcal{E}. \tag{7}$$

The model (7) can be written compactly as:

$$\rho(t+1) = F(\rho(t), u(t), w(t)), \tag{8}$$

where $F : \mathcal{P} \times \mathcal{U} \times \mathcal{W} \to \mathcal{P}$. Note that $F$ represents a hybrid piecewise affine dynamical system.

## III. Problem Formulation

In most cities, a periodic traffic light sequence, referred to as a 'fixed-time schedule', is predetermined. We denote a fixed-time schedule by a repeating sequence of control inputs with period $T$, $u_{(f)} := \overline{u_{(f)}^0, \ldots, u_{(f)}^{T-1}}$, where the over-line denotes cyclicity, i.e.,

$$u_f(t) = u_{(f)}^{t \bmod T}. \tag{9}$$

We make the following assumption about the design of the fixed-time schedule.

*Assumption* 1 (Properties of the fixed-time schedule, [14]). Given (8), there exists a fixed-time schedule $u(t) = u_f(t)$ and a set of sufficiently small disturbances $w(t) \in \mathcal{W}_s$, where $\mathcal{W}_s = \{w(t) : w(t) \preceq w_s^*, w_s^* \in \mathbb{R}^N, \forall t \in \mathbb{N}\}$, such that

(i) there exists a periodic orbit $\rho_f(t) := \overline{\rho_f^0, \ldots, \rho_f^{T-1}}$, where $\rho_f(t+1) = F(\rho_f(t), u_f(t), w^*)$, and,

(ii) for all initial conditions $\rho(0) \preceq \rho_f(0) + \epsilon$, where $\epsilon = [\epsilon_1 \cdots \epsilon_N]'$, applying (9) to the system results in state trajectories that are attracted to the lower-set of the periodic orbit $\rho_f(t)$, that is, $\forall \delta_i \in [0, \epsilon_i], \exists T_{\delta_i} \in \mathbb{N}, \epsilon_i \in \mathbb{R}_+$ such that $\rho_i(t) \leq \rho_{f,i}(t) + \delta_i, \forall t \geq T_{\delta_i}$, where $\rho_{f,i}$ is the $i$-th component of $\rho_f(t)$.

Assumption 1-(i) holds only if (8) is monotone with respect to the state and disturbances [14]. We will later show that (8) is indeed monotone. Assumption 1-(ii) states that there exists a margin of attraction $\epsilon$ for the periodic orbit $\rho_f(t)$. The design of $u_f(t)$ satisfying Assumption 1 has been detailed in [14, Section V].

We say that a state $\rho_i(t)$ is in the $\epsilon$-*vicinity* of the periodic orbit $\rho_f(t)$ at time $t$ if $\rho_i(t) \leq \rho_{f,i}(t) + \epsilon_i$. We say that the system is in the $\epsilon$-vicinity of the periodic orbit at time $t$ if $\rho(t) \leq \rho_f(t) + \epsilon$. Assumption 1 implies that small disturbances $\mathcal{W}_s = \{w(t) : w(t) \preceq w_s^*, w_s^* \in \mathbb{R}^N, \forall t \in \mathbb{N}\}$ do not take the state outside the $\epsilon$-vicinity of $\rho_f(t)$. Sometimes, larger disturbances due to demand surges or unmodelled driver behavior may cause the system to leave the $\epsilon$-vicinity of the periodic orbit. These disturbances typically last only for short periods of time and are concentrated in a particular cell. We refer to this class of disturbances, $w(t) \in \mathcal{W}_l$, where $\mathcal{W}_l = \mathcal{W} \backslash \mathcal{W}_s = \{w(t) : w_s^* \preceq w(t) \preceq w^*\}$, as large disturbances and restrict them to the following definition for the remainder of this paper.

*Definition* 2 (Large Disturbance). A large disturbance $w_{jam}^j(t) \in \mathcal{W}_l$ acting on cell $j$ between times $t_{jam} \in \mathbb{N}$ and $t_{end} \in \mathbb{N}$ is defined as $w_{jam}^j(t) = [0, \cdots, w_j(t), \cdots, 0]' \in \mathcal{W} \backslash \mathcal{W}_s$, with

$$w_j(t) = \begin{cases} \rho_m^j(s(t_{end}) - s(t_{mid})), & t_{mid} \leq t \leq t_{end}, \\ \rho_0^j(s(t_{mid}) - s(t_{jam})), & t_{jam} \leq t \leq t_{mid} < t_{end}, \\ 0, & \text{otherwise}, \end{cases} \quad (10)$$

where $w_{s,j}^* \leq \rho_0^j \leq w_j^*$ and $w_{s,j}$, $w_j^*$ represent the $j$-th components of the $w_s^*$ and $w^*$ vectors respectively and $s(\cdot)$ is the unit step function. The disturbance starts off at a higher magnitude $\rho_0^j$ during time $t_{jam} \leq t \leq t_{mid}$ and reduces to $\rho_m^j \leq \rho_0^j$ from time $t_{mid} \leq t \leq t_{end}$.

Let a large disturbance (10) act on cell $j$, causing the system to leave the $\epsilon$-vicinity of the periodic orbit. Such an excursion of the system trajectories from the $\epsilon$-vicinity of the desired periodic orbit is referred to as a *jam*, and the cell $j$ is referred to as the *origin of the jam*. We would like to use controllers in the local region around cell $j$ to return the states in that region to the $\epsilon$-vicinity of the periodic orbit within $T_d$ time steps. We would also like to ensure that the large disturbance does not spread beyond the local region, that is, the states outside the local region remain in $\epsilon$-vicinity of the periodic orbit. To this end, we first formally characterize the local region of a cell.

*Definition* 3 (Local Region). The local region of radius $d$ around cell $j$ is defined as

$$\mathcal{L}_{(j,d)} = \{l | dist_A(j \to l) \leq d\}, \quad (11)$$

where $A$ is the adjacency matrix defined in (2) and $dist_A(k \to j) := \min_{i \in \mathbb{N}} \left(A^i\right)_{jk} \neq 0$. The boundary of the local region $\mathcal{B}_{(j,d)} \subset \mathcal{L}_{(j,d)}$ is defined as $\mathcal{B}_{(j,d)} = \{l | dist_A(j \to l) = d\}$. We also define the set of incoming links at the boundary of the local region as $\mathcal{B}_{(j,d)}^{in} = \{l \in \mathcal{E}_i^{in} : i \in \mathcal{B}_{(j,d)}, l \notin \mathcal{L}_{(j,d)}\}$. Similarly we define the set of outgoing links at the boundary of the local region as $\mathcal{B}_{(j,d)}^{out} = \{l \in \mathcal{E}_i^{out} : i \in \mathcal{B}_{(j,d)}, l \notin \mathcal{L}_{(j,d)}\}$.

For cell $j$, define the local state vector $\rho_{(j,d)}(t)$ to be the vector of all $\rho_l(t)$ such that $l \in \mathcal{L}_{j,d}$. Similarly, define the local control vector and periodic orbit of cell $j$ as $u_{(j,d)}(t)$ and $\rho_{f_{(j,d)}}(t)$ respectively.

*Definition* 4 (Localizability). System (8) is said to be $(d, T_d)$-localizable, if, for a large disturbance (10) acting on cell $j$, $j \in \{1, \ldots, N\}$ and for all $t_{jam} \in \mathbb{N}$, there exists a control policy $u_{(j,d)}(t)$, $t_{jam} \leq t \leq T_d$, such that

(i) state trajectories $\rho_{j,d}(t) \in \mathcal{L}_{(j,d)}$ satisfy $\rho_{(j,d)}(t) \leq \rho_{f,(j,d)}(t)$ for all $t \geq T_d$,

(ii) state trajectories at the boundary, $\rho_l(t)$ such that $l \in \mathcal{B}_{j,d}$, satisfy $\rho_l(t) \leq \rho_{f,l}(t) + \epsilon$ for all $t$, and,

(iii) all states outside the local region $\mathcal{L}_{(j,d)}$ remain in the $\epsilon$-vicinity of the periodic orbit for all $t$.

With these definitions, we now formulate the problem addressed in this paper.

**Problem 1.** *Given system (8) with initial condition $\rho(0) \leq \rho_f(0)$, a fixed-time schedule $u_f(t)$ satisfying Assumption 1 and a disturbance $w(t) = w_{jam}^j(t)$ as defined in (10), find a control strategy $u_{(j,d)}(t)$ that only uses control inputs in the local region $\mathcal{L}_{(j,d)}$ to render the system $(d, T_d)$-localizable.*

## IV. CONTROL DESIGN

In this section, we propose a MILP formulation to solve Problem 1. We first state a property of the system that will be useful in control design.

**Theorem 1** (Monotonicity)**.** System (8) is monotone with respect to the state $\rho$ and disturbances $w$ in the sense that

$$\begin{aligned} \rho \preceq \rho' &\Rightarrow F(\rho, u, w) \preceq F(\rho', u, w), \forall u \in \mathcal{U}, \forall w \in \mathcal{W} \\ w \preceq w' &\Rightarrow F(\rho, u, w) \preceq F(\rho, u, w'), \forall u \in \mathcal{U}, \forall w, w' \in \mathcal{W} \end{aligned} \quad (12)$$

The proof of Theorem 1 is omitted due to space constraints. We now formulate the set of constraints that should be incorporated into the MILP to ensure localizability.

### A. Localizability

System (8) must satisfy constraints **C1-C5** detailed below to ensure $(d, T_d)$-localizability according to Definition 4.

**C1:** The evolution of the system satisfies

$$\rho(t+1) \preceq F(\rho(t), u(t), w^*), \forall t \in \mathbb{N}. \quad (13)$$

**C2:** The system densities are within the physically admissible range, i.e., $\rho(t) \in \mathcal{P}, \forall t$.

**C3:** Boundary inflow: The density of vehicles in all incoming cells at the boundary of the local region satisfies

$$\rho_l(t) \leq \rho_{f,l}(t) + \epsilon_{\mathcal{E}_l^{in}}, \forall l \in \mathcal{B}_{(j,d)}^{\text{in}}, \forall t \in [0, T_d], \quad (14)$$

where $\epsilon_{\mathcal{E}_l^{\text{in}}} = \min_{i \in \mathcal{E}_l^{\text{in}}} \epsilon_i$. This constraint ensures that the loss in the vacancy available in cell $l$ does not affect the flow from upstream by more than its margin of attraction.

**C4:** Boundary outflow: The numbers of vehicles that flow out of all outgoing links at the boundary of the local region remain in the $\epsilon$-vicinity of their baseline flows, that is,

$$\sum_{k \in \mathcal{E}_l^{\text{out}}} Q_{l,k}(t) \leq \sum_{k \in \mathcal{E}_l^{\text{out}}} Q_{f_{(l,k)}}(t) + \epsilon_{\mathcal{E}_l^{\text{out}}}, \quad \forall l \in \mathcal{B}_{(j,d)}^{\text{out}}, \forall t \in [0, T_d], \tag{15}$$

where $Q_{f_{(l,k)}}(t)$ is the flow from cell $l$ to cell $k$ for the periodic orbit resulting from the fixed-time schedule and $\epsilon_{\mathcal{E}_l^{\text{out}}} = \min_{i \in \mathcal{E}_l^{\text{out}}} \epsilon_i$. This constraint ensures that the outflow from the local region does not increase beyond the margin of attraction of the cells outside $\mathcal{B}_{(j,d)}$.

**C5:** The state in the local region reaches the interior of the lower-set of $(\rho_{f_{(j,d)}}(t) + \epsilon_j)$ at time $T_d$, that is,

$$\rho_i(T_d) \leq \rho_{f_{i,(j,d)}}(T_d) + \epsilon_i, \forall i \in \mathcal{L}_{(j,d)} \setminus (\mathcal{B}_{(j,d)}^{\text{out}} \cup \mathcal{B}_{(j,d)}^{\text{in}}), \tag{16}$$

where $\rho_{f_{i,(j,d)}}$ is the $i$-th component of $\rho_{f_{(j,d)}}$.

Constraints **C1-C2** encapsulate the dynamics of the system with all possible mode switchings and the effect of the discrete control inputs on the system. Due to the monotonicity of the system, enforcing **C1** on the dynamics corresponding to the maximum possible disturbance automatically ensures that it holds for all other $w(t)$. Constraints **C3** and **C4** enforce the condition in Definition 4-(ii), by requiring that the flows at the boundary of the local region remain unaffected by the disturbance, ensuring that the disturbance does not propagate beyond the local region. Constraint **C5** states that the states in the local region return to the $\epsilon$-vicinity of the periodic orbit in $T_d$ time steps, as required in Definition 4-(i) for localizability. We now show, in Theorem 2, that a control sequence which when applied to (8) results in state trajectories that satisfy constraints **C1-C5** renders the system $(d, T_d)$-localizable.

**Theorem 2.** System (8) is $(d, T_d)$-localizable if and only if, for every $j \in \mathcal{E}$ and $w(t) = w_{jam}^j(t)$, there exists an open-loop control sequence

$$u_j^{local,T_d} := u_{(j,d)}(0), u_{(j,d)}(1), \cdots, u_{(j,d)}(T_d), \tag{17}$$

such that the resulting response of the system for time $t_{jam} \leq t \leq T_d$ satisfies constraints **C1-C5**.

*Proof.* **(Necessity)** Let the initial conditions be $\rho(0) = \rho_f(0)$ and $w(0) = w^* + w_{jam}^j(0)$ (maximal disturbances). Clearly, there must exist at least one trajectory $\rho_{(j,d)}(0), \rho_{(j,d)}(1), \cdots, \rho_{(j,d)}(T_d)$ satisfying the constraints **C1-C5** for which we obtain the control sequence $u_{(j,d)}(0), u_{(j,d)}(1), \cdots, u_{(j,d)}(T_d)$.

**(Sufficiency)** For the general case, we have $\rho(t) \leq \rho_f(t)$ and $w(t) \leq w^* + w_{jam}^j(t)$. By applying the same $u_{(j,d)}(0), u_{(j,d)}(1), \cdots, u_{(j,d)}(T_d)$ in an open-loop manner, we obtain the trajectory $\rho'_{(j,d)}(0), \rho'_{(j,d)}(1), \cdots, \rho'_{(j,d)}(T_d)$. From Theorem 1, due to the monotonicity of (8), we have $F(\rho'_{(j,d)}(\tau), u_{(j,d)}(\tau), w) \preceq F(\rho_{(j,d)}(\tau), u_{(j,d)}(\tau), w^*), \tau = 0, 1, \cdots, T_d$. Hence $\rho'_{(j,d)}(\tau) \preceq \rho_{(j,d)}(\tau)$, $\tau = 0, 1, \cdots, T_d$. Therefore, the trajectory $\rho'_{(j,d)}(0), \rho'_{(j,d)}(1), \cdots, \rho'_{(j,d)}(T_d)$ of length $(T_d + 1)$ also satisfies **C1-C5**. □

## B. Control Synthesis

The computation of the localizing control sequence $u^{local,T_d}$ is formulated as the following constraint satisfaction problem:

$$P_1: \text{ find } u_j^{local,T_d}, \quad \text{subject to } \textbf{C1-C5}, \tag{18}$$

which is then rewritten as a MILP. The solution to the optimization problem $P_1$ is the control sequence for which the system is $(d, T_d)$-localizable with the maximum disturbance, that is, $\rho_j^0 = \rho_m^0 = w^*$ in (10). Due to the monotonicity of the system, this control sequence, designed for the largest disturbance, is sufficient to localize any smaller disturbance.

We note that constraints **C2-C5** are linear, and the only integer constraints are those that result from **C1**. The constraint **C1** represents the system dynamics, which capture discrete control inputs and determine the system mode.

Based on Proposition 2 from [15], constraint **C1** can be formulated as a set of mixed integer constraints. The piecewise affine hybrid dynamics in (8) can be written as

$$\begin{aligned} &\rho(t+1) = A\rho(t) + E_u u(t) + E_w w(t) + E_\beta \beta(t) + E_z z(t) \\ &E_\beta \beta(t) + E_z z(t) \leq E_\rho \rho(t) + E_u u(t) + e \end{aligned} \tag{19}$$

where $\beta(t) \in \{0, 1\}^{n_b}$ and $z(t) \in \mathbb{R}^{n_r}$ are auxiliary variables, $E_\rho$, $E_u$, $E_w$, $E_r$ and $E_\delta$ are appropriately sized constant matrices, and $e$ is a constant vector which makes the set of mixed-integer constraints well posed in the sense that the feasible set of $\rho(t+1)$ is a single point equal to $F(\rho(t), u(t), w(t))$.

From (19) and the linearity of constraints **C2-C5**, we can see that $P_1$ is a MILP. Further, from Theorem 2, it is easy to see that the feasibility of the MILP $P_1$ for every $j \in \mathcal{E}$ is necessary and sufficient for $(d, T_d)$-localizability of (8). Algorithm 1 describes the design of the localizing controller based on the MILP formulation.

**Algorithm 1** Control Design Procedure

1: Design a fixed-time control strategy satisfying Assumption 1 according to the procedure laid out in [14] by solving a MILP over all cells in the network. This computation is carried out only once offline.
2: For every node $j$, determine the smallest possible $d$ and $T_c$ for which the MILP $P_1$ is feasible.
3: Determine the control sequence $u_j^{local,T_d}$ corresponding to disturbance $w_j^*$ for every cell $j$.

## C. Controller Implementation

We assume that a suitably calibrated binary detector that determines whether a large disturbance has occurred at every time instant exists at each link. Such a detector can easily be implemented using cameras and calibrated using the $\epsilon$-margin of the system. When a large disturbance is detected at cell $j$, we switch to the precomputed control sequence $u_j^{local,T_d}$ determined in Algorithm 1 in the local region $\mathcal{L}_{(j,d)}$. When the system trajectories in the local region return to the $\epsilon$-vicinity of the periodic orbit, the fixed-time schedule is resumed.

### D. Discussion

We make the following observations about the proposed control strategy.

- Our implementation allows for an open-loop controller implementation without any online computation. Since the system is monotone, a controller designed to localize the worst-case disturbance $w^*$ can localize any smaller disturbance. Therefore, the only feedback that is necessary for our implementation is the location of the disturbed cell. However, we note that arbitrarily large disturbances are obviously not localizable with finite capacity roads.
- The implementation uses controllers from the smallest possible radius $d$ for which the MILP is feasible. We note that the computation time of MILP feasibility checking grows exponentially with respect to the number of integers. Therefore, the design is only efficient when the system has a small localization radius.

## V. Case Study

In this section, we present a case study of an urban traffic network to demonstrate the control policy proposed in Section IV. We consider the network shown in Fig. 2, which comprises of a grid of 112 roads (cells) and 49 intersections. Each cell is assumed to have a capacity of 40 vehicles and a maximum outflow of 15 vehicles. The inflows are assumed to be $\lambda = 4$ in the East/South entries, $\lambda = 3$ in West/South Entries and $\lambda = 3$ in West/North Entries. The turning ratios for vehicles heading straight and turning (left or right) are assumed to be 0.9 and 0.1 respectively and the capacity ratios of all cells are assumed to be 1. The free and congested speeds for all cells are set to 1 per time step.

For this network, we first design a fixed-time schedule of periodicity 2 using the procedure in [14] and obtain the margin of attraction $\epsilon$ for 98 cells (not including the 14 cells with flows exiting the network). For cell $j$ indicated in Fig. 2, we find that the $\epsilon$-margin is 4.4084. We then design the local controllers for every cell in the network by solving the MILP (18) using Gurobi[1] and obtain a localization radius of $d = 2$ and localization time of $T_d = 10$. The computation of the solution to the MILP (18) for this network required about 3 seconds on a 2.7 GHz dual-core iMac. We test the localized control policy by applying the disturbance shown in Fig. 2 to the system. When the disturbance is applied, we switch to the local control sequences computed by solving the MILP (18) in the local region indicated in Fig. 2. After $T_d = 10$ cycles we switch back to the fixed-time schedule in the local region. We enforce the fixed-time schedule on all cells outside the local region throughout the simulation. Fig. 6 shows the evolution of the system states under the

[1] http://www.gurobi.com/

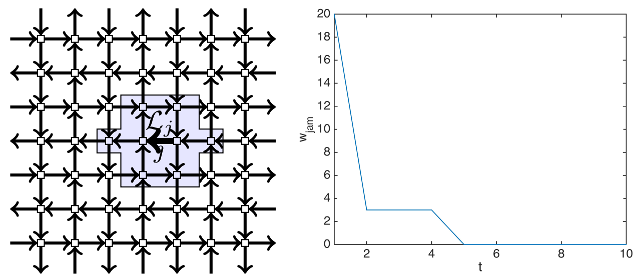


Fig. 2. (Left) Urban traffic network indicating cell $j$ and its local region $\mathcal{L}_j$ with $d = 2$ (shaded); (Right) Large disturbance signal.

following simulation conditions.

1) No disturbance applied: The system trajectories converge to a periodic orbit starting from an appropriate initial condition, which agrees with Assumption 1 and the discussion in [14].
2) With fixed-time non-local control policy: The system returns to the $\epsilon$-vicinity of the periodic orbit in 11 cycles.
3) With localized control policy: The disturbance is localized to a radius $d = 2$ (shown in Fig. 2 around cell $j$ and the system returns to the $\epsilon$-vicinity of the periodic orbit in 9.5 cycles.

We observe that applying the localized control policy prevents the propagation of the jam beyond a radius of $d = 2$ and ensures that the system trajectories return to the $\epsilon$-vicinity of the periodic orbit faster. Fig. 4 and Fig. 5 show snapshots of the spatial propagation of the jam in the network at various simulation times with the local and fixed-time control policy respectively. It is observed while both control policies are eventually able to attenuate the disturbance, the localized policy does it more effectively by not allowing it to spread greatly to other cells. This behavior is more clearly seen in the trajectories in Fig. 6.

While we observe that disturbances in (8) can be localized with discrete control inputs, the performance of the localization scheme is limited due to the small number of sparse actuators (traffic lights) available in the system. In other words, the system is heavily under-actuated. In our simulations, we were not able to find a case where the local control policy was able to localize a disturbance while the use of the fixed-time schedule instead would have caused the network to be jammed. We also note that employing a well-designed centralized control policy can sometimes result in a better performance than the local control policy.

We believe that better localization performance can be achieved through the use of continuous actuation like variable speed limits in addition to discrete traffic signals. The designs in this paper can also be extended to optimize performance objectives like network throughput and average travel time by adding a cost function to problem $P_1$. These directions which will be the subject of future work.

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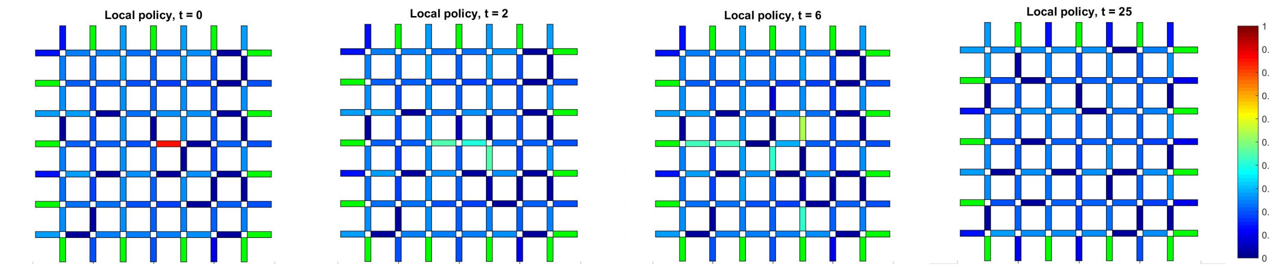


Fig. 3. Spatial propagation of jam with local control policy. Color map: blue to red - lower to higher density, 0: zero vehicles, 1: 30 vehicles.

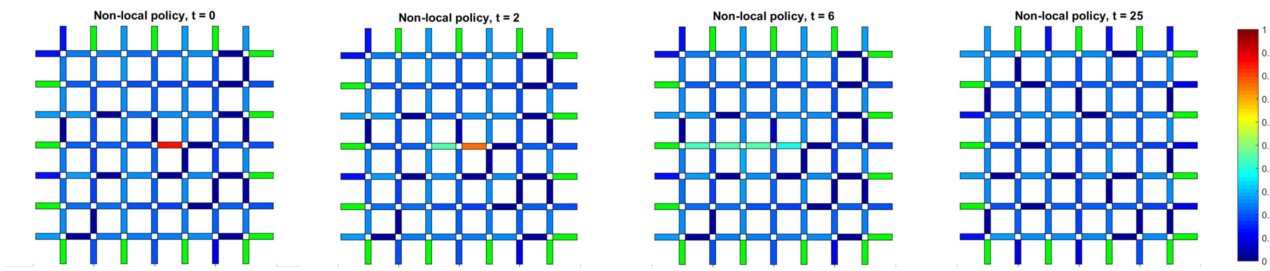


Fig. 4. Spatial propagation of jam with non-local control policy. Color map: blue to red - lower to higher density, 0: zero vehicles, 1: 30 vehicles.

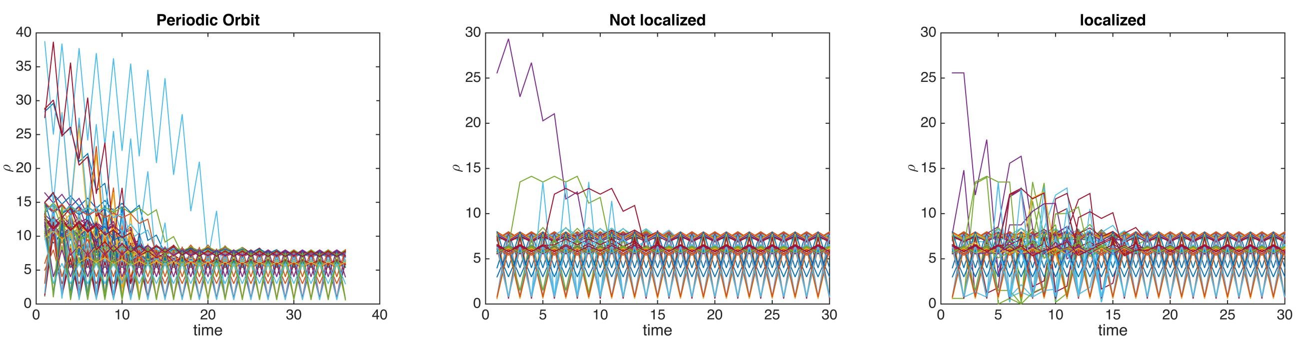


Fig. 5. Plot of system trajectories with respect to time. (Left) The trajectory of the fixed-time schedule converges to a periodic orbit, where $\epsilon$ is found from the minimal difference between the initial $\rho$ and the periodic $\rho_f$. (Middle): System trajectories with large disturbance from Fig. 2 under non-local fixed-time control policy, (Right) System trajectories with large disturbance from Fig. 2 under localized control policy.


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